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On the theory of elastic shells made from a material with voids

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Abstract

In this paper we present a theory for porous elastic shells using the model of Cosserat surfaces. We employ the Nunziato–Cowin theory of elastic materials with voids and introduce two scalar fields to describe the porosity of the shell: one field characterizes the volume fraction variations along the middle surface, while the other accounts for the changes in volume fraction along the shell thickness. Starting from the basic principles, we first deduce the equations of the nonlinear theory of Cosserat shells with voids. Then, in the context of the linear theory, we prove the uniqueness of solution for the boundary initial value problem. In the case of an isotropic and homogeneous material, we determine the constitutive coefficients for Cosserat shells, by comparison with the results derived from the three-dimensional theory of elastic media with voids. To this aim, we solve two elastostatic problems concerning rectangular plates with voids: the pure bending problem and the extensional deformation under hydrostatic pressure.

Keywords: Cosserat shells and plates; Elastic materials with voids; Porous; Extensional; Bending; Constitutive coefficients

1. Introduction

In this paper, we employ the Nunziato–Cowin theory for elastic materials with voids in order to describe the mechanical behavior of porous shells. For this purpose, we consider thin elastic shells modeled as Cosserat surfaces.

Nunziato and Cowin (1979) and Cowin and Nunziato (1983) have elaborated a theory for the treatment of porous solids in which the matrix material is elastic and the interstices are void of material. This theory is aimed to describe the behavior of geological and manufactured porous bodies, as well as granular materials. Thus, using an idea first presented by Goodman and Cowin (1972), the bulk density of the porous body

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ρ is written as the product of two fields: the matrix material density field γ and the volume fraction field v, i.e.,

$$\rho = \gamma v \quad (0 < v \leqslant 1).$$

In this way, the volume fraction field represents a kinematical variable assigned to each material particle. The interpretation of this field from the theory of media with microstructure viewpoint was given by Capriz and Podio-Guidugli (1981) and Capriz (1989). The Nunziato–Cowin theory has been used in many works to investigate the behavior of deformable porous bodies (see, e.g., Puri and Cowin, 1985; Ciarletta and Ieşan, 1993; Bîrsan, 2003a).

Our approach to the theory of elastic shells is based on the theory of Cosserat surfaces. This theory utilizes a two-dimensional model for the shell, consisting in a surface together with a deformable vector (called *director*) assigned to every point. The foundations of the theory of Cosserat surfaces are discussed in the work of Naghdi (1972), where a comparison with classical theories of shells is also presented. A modern approach to the theory of Cosserat shells and several of its applications can be found in the books of Antman (1995) and Rubin (2000).

A theory for Cosserat shells with voids was first presented by Bîrsan (2000a,b) and was employed recently to determine the solution of Saint-Venant's problem for porous cylindrical shells (see Bîrsan, 2005). This theory accounts for the variations of the volume fraction field along the middle surface of the shell, but not for the changes in porosity along the shell thickness. The main purpose of the present paper is to establish a theory of Cosserat shells with voids which allows, in addition, for the characterization of the volume fraction variations along the thickness of the shell. To this aim, we introduce two volume fraction fields associated to every point of the Cosserat shell: one field describes the changes in porosity along the middle surface, while the other takes account of the porosity variations along the thickness of the shell. The former can be interpreted as the average volume fraction through the thickness of the three-dimensional shell, while the latter represents the average (through-the-thickness) volume fraction gradient.

We begin our study by presenting the basic principles and deducing the governing equations for the nonlinear theory of Cosserat shells with voids. In Section 3, we confine our attention to the linear theory. In this context, we formulate the boundary initial value problem and we prove the uniqueness of its solution, without any definiteness assumptions on the internal energy of the shell. In Section 4, we deduce the constitutive equations for Cosserat shells and plates made from isotropic and homogeneous materials with voids. Then, we remark the uncoupling between the equations governing the extensional motions and the bending deformations of plates with voids. We show that the same equations for plates can be derived starting from the three-dimensional theory of elastic media with voids. Thus, the bending equations for plates with voids (obtained here by direct approach) have been studied previously in the context of the theory of Mindlin-type plates by Scarpetta (2002) and Bîrsan (2003b, in press). In Section 5, we identify the constitutive coefficients for isotropic Cosserat shells, by comparison of certain simple solutions with corresponding exact solutions in the three-dimensional theory of elastic materials with voids. In this order, we consider two elastostatic problems: the pure bending of a rectangular plate and the extensional deformation of a plate under hydrostatic pressure.

2. Principles and governing equations

We begin this section by summarizing the main kinematics for the theory of Cosserat shells. Then, we postulate the basic principles and we derive the field equations that govern the mechanics of elastic shells with voids.

By definition, a Cosserat shell is a body $\mathscr{C}$ comprising a two-dimensional surface embedded in a Euclidean 3-space and a single director (that is, deformable vector) attached to every point of the surface. We refer to the monograph of Naghdi (1972) for a detailed analysis of this model.

We consider a Cosserat surface $\mathscr{C}$ which particles are identified by the curvilinear material coordinates θ^α ($\alpha = 1,2$) and we denote by $\mathscr{S}_0$ and $\mathscr{S}$ the surface of $\mathscr{C}$ in the reference configuration and in the present configuration at time t, respectively. Let $\boldsymbol{r}$ and $\boldsymbol{d}$ designate, respectively, the position vector of a typical point of $\mathscr{S}$ relative to a fixed origin and the director at $\boldsymbol{r}$. Then, a motion of the Cosserat shell is defined by

$$\boldsymbol{r} = \boldsymbol{r}(\theta^\alpha, t), \qquad \boldsymbol{d} = \boldsymbol{d}(\theta^\alpha, t), \tag{2.1}$$

and we assume that $\boldsymbol{d}$ is nowhere tangent to $\mathscr{S}$.

We denote by $\boldsymbol{a}_\alpha(\theta^\alpha, t)$ the covariant base vectors along the θ^α-curves on $\mathscr{S}$, by $\boldsymbol{a}_3(\theta^\alpha, t)$ the unit normal to $\mathscr{S}$, and by $a_{\alpha\beta}$ and $b_{\alpha\beta}$ the first and the second fundamental forms of the surface $\mathscr{S}$, respectively. We have

$$\boldsymbol{a}_\alpha = \frac{\partial \boldsymbol{r}}{\partial \theta^\alpha}, \qquad \boldsymbol{a}_3 = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}, \qquad (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{d}) > 0, \qquad a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \qquad b_{\alpha\beta} = \boldsymbol{a}_3 \cdot \boldsymbol{a}_{\alpha,\beta}, \tag{2.2}$$

where the comma preceding a subscript denotes partial differentiation with respect to the corresponding coordinate. We also employ the notations

$$d_i = \boldsymbol{d} \cdot \boldsymbol{a}_i, \qquad \lambda_{i\alpha} = \boldsymbol{d}_{,\alpha} \cdot \boldsymbol{a}_i. \tag{2.3}$$

Throughout this paper, Latin indices take the values $\{1, 2, 3\}$, while Greek indices are confined to the range $\{1, 2\}$. The usual summation convention is also used.

We identify the reference surface $\mathscr{S}_0$ with the initial surface and we designate by $\boldsymbol{R} = \boldsymbol{R}(\theta^\alpha)$ the position vector, $\boldsymbol{D} = \boldsymbol{D}(\theta^\alpha)$ the reference director at $\boldsymbol{R}$, and let $\boldsymbol{A}_\alpha$ and $\boldsymbol{A}_3$ be, respectively, the base vectors along the θ^α-curves on $\mathscr{S}_0$ and the unit normal to $\mathscr{S}_0$. Then

$$\begin{aligned} &\boldsymbol{A}_\alpha = \frac{\partial \boldsymbol{R}}{\partial \theta^\alpha}, \qquad \boldsymbol{A}_3 = \frac{\boldsymbol{A}_1 \times \boldsymbol{A}_2}{|\boldsymbol{A}_1 \times \boldsymbol{A}_2|}, \qquad A_{\alpha\beta} = \boldsymbol{A}_\alpha \cdot \boldsymbol{A}_\beta, \\ &B_{\alpha\beta} = \boldsymbol{A}_3 \cdot \boldsymbol{A}_{\alpha,\beta}, \qquad D_i = \boldsymbol{D} \cdot \boldsymbol{A}_i, \qquad \Lambda_{i\alpha} = \boldsymbol{D}_{,\alpha} \cdot \boldsymbol{A}_i. \end{aligned} \tag{2.4}$$

We define the relative kinematic measures by

$$e_{\alpha\beta} = \tfrac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}), \qquad \gamma_i = d_i - D_i, \qquad \varkappa_{i\alpha} = \lambda_{i\alpha} - \Lambda_{i\alpha}. \tag{2.5}$$

The velocity and the director velocity vectors are given by

$$\boldsymbol{v} = \dot{\boldsymbol{r}}, \qquad \boldsymbol{w} = \dot{\boldsymbol{d}}, \tag{2.6}$$

where a superposed dot stands for the material derivative with respect to t, holding θ^α fixed.

In order to describe the porosity of a Cosserat shell with voids, we consider two independent scalar fields $\nu(\theta^\alpha, t)$ and $\chi(\theta^\alpha, t)$ called the *volume fraction fields* ($0 < \nu \leqslant 1$). We mention that, in the previous approaches of the theory of Cosserat shells with voids (see Bîrsan, 2000a, 2005), only one porosity field has been admitted, corresponding to $\nu(\theta^\alpha, t)$, which permits for the characterization of the changes in volume fraction along the middle surface of the three-dimensional shell. In this paper, we introduce an additional porosity field, namely $\chi(\theta^\alpha, t)$, which will give us information about the volume fraction variations along the shell thickness. The significance of these two porosity fields will be clarified in Section 4.

The kinetic energy per unit mass of $\mathscr{C}$ in the present configuration has the expression

$$T = \tfrac{1}{2}\left(\boldsymbol{v} \cdot \boldsymbol{v} + \bar{\alpha} \boldsymbol{w} \cdot \boldsymbol{w} + \kappa_1 \dot{\nu}^2 + \kappa_2 \dot{\chi}^2\right), \tag{2.7}$$

where $\rho = \rho(\theta^\alpha, t)$ is the mass per unit area of $\mathscr{S}$ and the inertia coefficients $\bar{\alpha}$, κ_1, κ_2 are prescribed functions of θ^α and independent of t. Let $\epsilon = \epsilon(\theta^\alpha, t)$ be the internal energy measured per unit mass of $\mathscr{S}$.

We now postulate the basic principles which are assumed to hold for each part $\mathscr{P}$ of $\mathscr{S}$ and for every motion of the Cosserat shell with voids.

The mass conservation principle is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho \,\mathrm{d}\sigma = 0. \tag{2.8}$$

Balance of energy is stated in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho(T+\epsilon)\,\mathrm{d}\sigma = \int_{\mathscr{P}} \rho(\boldsymbol{f}\cdot\boldsymbol{v}+\boldsymbol{l}\cdot\boldsymbol{w}+p\dot{v}+P\dot{\chi})\,\mathrm{d}\sigma + \int_{\partial\mathscr{P}} (\boldsymbol{N}\cdot\boldsymbol{v}+\boldsymbol{M}\cdot\boldsymbol{w}+h\dot{v}+H\dot{\chi})\,\mathrm{d}s. \tag{2.9}$$

Here, $\boldsymbol{N}=\boldsymbol{N}(\theta^\alpha,t;\boldsymbol{n})$ is the force vector and $\boldsymbol{M}=\boldsymbol{M}(\theta^\alpha,t;\boldsymbol{n})$ is the director force vector (also called director couple) at the curve $\partial\mathscr{P}$, $\boldsymbol{n}=n_\alpha\boldsymbol{a}^\alpha$ is the outward unit normal to $\partial\mathscr{P}$ tangent to $\mathscr{S}$, while $\boldsymbol{f}=\boldsymbol{f}(\theta^\alpha,t)$ and $\boldsymbol{l}=\boldsymbol{l}(\theta^\alpha,t)$ stand for the assigned force and the assigned director force vectors, respectively (see Naghdi, 1972; Green and Naghdi, 1979). The scalar fields $p=p(\theta^\alpha,t)$ and $P=P(\theta^\alpha,t)$ are the assigned equilibrated body forces, while $h=h(\theta^\alpha,t;\boldsymbol{n})$ and $H=H(\theta^\alpha,t;\boldsymbol{n})$ are the equilibrated stresses acting on the curve $\partial\mathscr{P}$.

Principle of director momentum has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho\bar{\alpha}\boldsymbol{w}\,\mathrm{d}\sigma = \int_{\mathscr{P}} (\rho\boldsymbol{l}-\boldsymbol{m})\,\mathrm{d}\sigma + \int_{\partial\mathscr{P}} \boldsymbol{M}\,\mathrm{d}s, \tag{2.10}$$

where $\boldsymbol{m}$ is the internal director force vector.

Balance of equilibrated force (see Nunziato and Cowin, 1979; Goodman and Cowin, 1972) is assumed in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho\kappa_1\dot{v}\,\mathrm{d}\sigma = \int_{\mathscr{P}} (\rho p-g)\,\mathrm{d}\sigma + \int_{\partial\mathscr{P}} h\,\mathrm{d}s, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho\kappa_2\dot{\chi}\,\mathrm{d}\sigma = \int_{\mathscr{P}} (\rho P-G)\,\mathrm{d}\sigma + \int_{\partial\mathscr{P}} H\,\mathrm{d}s, \tag{2.11}$$

where the scalar fields g and G stand for the internal equilibrated body forces.

We also assume that the invariance conditions under superposed rigid body motions hold, i.e., we require that all the above fields be objective. Using the same procedure as Naghdi (1972), from the balance of energy (2.9) and the invariance conditions under superposed rigid body motions we obtain the principle of momentum and the principle of moment of momentum, respectively, as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho\boldsymbol{v}\,\mathrm{d}\sigma = \int_{\mathscr{P}} \rho\boldsymbol{f}\,\mathrm{d}\sigma + \int_{\partial\mathscr{P}} \boldsymbol{N}\,\mathrm{d}s \tag{2.12}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho(\boldsymbol{r}\times\boldsymbol{v}+\bar{\alpha}\boldsymbol{d}\times\boldsymbol{w})\,\mathrm{d}\sigma = \int_{\mathscr{P}} \rho(\boldsymbol{r}\times\boldsymbol{f}+\boldsymbol{d}\times\boldsymbol{l})\,\mathrm{d}\sigma + \int_{\partial\mathscr{P}} (\boldsymbol{r}\times\boldsymbol{N}+\boldsymbol{d}\times\boldsymbol{M})\,\mathrm{d}s. \tag{2.13}$$

Under suitable continuity assumptions, from Eqs. (2.10)–(2.12) we obtain that the fields $\boldsymbol{N}$, $\boldsymbol{M}$, h and H can be expressed as

$$\boldsymbol{N}=\boldsymbol{N}^\alpha n_\alpha, \qquad \boldsymbol{M}=\boldsymbol{M}^\alpha n_\alpha, \qquad h=h^\alpha n_\alpha, \qquad H=H^\alpha n_\alpha. \tag{2.14}$$

Then, the local field equations corresponding to the above principles are

- equation of mass conservation

$$\dot{\rho}+\rho\boldsymbol{a}^\alpha\cdot\boldsymbol{v}_{,\alpha}=0; \tag{2.15}$$

- equation of momentum

$$\boldsymbol{N}^\alpha{}_{|\alpha}+\rho\boldsymbol{f}=\rho\dot{\boldsymbol{v}}; \tag{2.16}$$

- equation of director momentum

$$\boldsymbol{M}^{\alpha}{}_{|\alpha} - \boldsymbol{m} + \rho \boldsymbol{l} = \rho \bar{\alpha} \dot{\boldsymbol{w}}; \tag{2.17}$$

- equation of moment of momentum

$$\boldsymbol{a}_{\alpha} \times \boldsymbol{N}^{\alpha} + \boldsymbol{d} \times \boldsymbol{m} + \boldsymbol{d}_{,\alpha} \times \boldsymbol{M}^{\alpha} = \boldsymbol{0}; \tag{2.18}$$

- equations of equilibrated force

$$h^{\alpha}{}_{|\alpha} - g + \rho p = \rho \kappa_1 \ddot{v}, \qquad H^{\alpha}{}_{|\alpha} - G + \rho P = \rho \kappa_2 \ddot{\chi}; \tag{2.19}$$

- balance of energy

$$\rho \dot{\epsilon} = \boldsymbol{N}^{\alpha} \cdot \boldsymbol{v}_{,\alpha} + \boldsymbol{M}^{\alpha} \cdot \boldsymbol{w}_{,\alpha} + \boldsymbol{m} \cdot \boldsymbol{w} + g \dot{v} + G \dot{\chi} + h^{\alpha} \dot{v}_{,\alpha} + H^{\alpha} \dot{\chi}_{,\alpha}. \tag{2.20}$$

In the above equations the subscript vertical bar stands for covariant differentiation with respect to the metric tensor $a_{\alpha\beta}$.

In what follows, we present the constitutive equations for Cosserat shells with voids and we deduce the restrictions imposed on these equations by the balance of energy relation (2.20), using the procedure described by Naghdi (1972, Section 14β).

We assume that the variables

$$\epsilon, \boldsymbol{N}^{\alpha}, \boldsymbol{M}^{\alpha}, \boldsymbol{m}, g, G, h^{\alpha}, H^{\alpha}$$

are functions of the set

$$\boldsymbol{a}_{\alpha}, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}, v, \chi, v_{,\beta}, \chi_{,\beta}$$

and, in addition, depend also on the particle θ^{μ}.

Then, the energy equation (2.20) is identically satisfied for all arbitrary values of $\dot{\boldsymbol{a}}_{\alpha}$, $\dot{\boldsymbol{d}}$, $\dot{\boldsymbol{d}}_{,\gamma}$, $\dot{v}$, $\dot{\chi}$, $\dot{v}_{,\beta}$ and $\dot{\chi}_{,\beta}$ provided we have

$$\begin{aligned}
&\epsilon = \hat{\epsilon}\left(\boldsymbol{a}_{\alpha}, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}, v, \chi, v_{,\beta}, \chi_{,\beta}; \theta^{\mu}\right),\\
&\boldsymbol{N}^{\alpha} = \rho \frac{\partial \hat{\epsilon}}{\partial \boldsymbol{a}_{\alpha}}, \qquad \boldsymbol{M}^{\alpha} = \rho \frac{\partial \hat{\epsilon}}{\partial \boldsymbol{d}_{,\alpha}}, \qquad \boldsymbol{m} = \rho \frac{\partial \hat{\epsilon}}{\partial \boldsymbol{d}},\\
&g = \rho \frac{\partial \hat{\epsilon}}{\partial v}, \qquad h^{\alpha} = \rho \frac{\partial \hat{\epsilon}}{\partial v_{,\alpha}}, \qquad G = \rho \frac{\partial \hat{\epsilon}}{\partial \chi}, \qquad H^{\alpha} = \rho \frac{\partial \hat{\epsilon}}{\partial \chi_{,\alpha}}.
\end{aligned} \tag{2.21}$$

The constitutive equations (2.21) can be further restricted by imposing the invariance conditions under superposed rigid body motions. Thus, using the same method as Naghdi (1972, Section 13), we can write the constitutive relations $(2.21)_{1-4}$ in an alternative form, expressed in terms of tensor components.

To recapitulate, the governing equations of the nonlinear theory of Cosserat shells with voids are the mass conservation equation (2.15), the equations of motion (2.16)–(2.18), the equations of equilibrated force (2.19) and the constitutive equations (2.21).

We remark that Eqs. (2.15)–(2.18) have the same form as the mass conservation equation and the equations of motion from the theory of Cosserat shells (without voids). These equations can be written with the help of tensor components, as presented by Naghdi (1972).

3. Linear theory

In this section, we deduce the equations of the linear theory of Cosserat shells with voids. Then, we formulate the boundary initial value problem and we prove the uniqueness of solution.

In the linear theory, we assume that

$$\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{u}, \qquad \boldsymbol{d} = \boldsymbol{D} + \boldsymbol{\delta}, \qquad v = v_0 + \varphi, \qquad \chi = \chi_0 + \psi \tag{3.1}$$

and

$$\boldsymbol{u} = \varepsilon \boldsymbol{u}' = u^i \boldsymbol{A}_i = u_i \boldsymbol{A}^i, \qquad \boldsymbol{\delta} = \varepsilon \boldsymbol{\delta}' = \delta^i \boldsymbol{A}_i = \delta_i \boldsymbol{A}^i, \qquad \varphi = \varepsilon \varphi', \qquad \psi = \varepsilon \psi', \tag{3.2}$$

where ε is a *small* non-dimensional parameter. In (3.1), v_0 and χ_0 represent the values of the volume fraction fields v and χ in the reference configuration.

Let us confine our attention hereafter to Cosserat shells of uniform thickness. This case is characterized by the relation

$$\boldsymbol{D} = \boldsymbol{A}_3. \tag{3.3}$$

We also assume that v_0 is constant and $\chi_0 = 0$.

Following the classical linearization procedure (see, e.g., Naghdi, 1972; Green and Naghdi, 1979) we obtain the expressions for the kinematic measures defined in (2.5) in the forms

$$\begin{aligned} &e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - B_{\alpha\beta} u_3, \qquad \gamma_\alpha = \delta_\alpha + u_{3,\alpha} + B^\gamma_\alpha u_\gamma, \qquad \gamma_3 = \delta_3, \\ &\varkappa_{\beta\alpha} = \delta_{\beta|\alpha} - B_{\alpha\beta}\delta_3 - B^\gamma_\alpha u_{\gamma|\beta} + B^\gamma_\alpha B_{\beta\gamma} u_3, \qquad \varkappa_{3\alpha} = \delta_{3,\alpha} + B^\gamma_\alpha \delta_\gamma + B^\gamma_\alpha u_{3,\gamma} + B^\beta_\alpha B^\gamma_\beta u_\gamma. \end{aligned} \tag{3.4}$$

In (3.4) the subscript vertical bar denotes covariant differentiation with respect to $A_{\alpha\beta}$.

For the linear theory, we assume that $\boldsymbol{N}^\alpha$, $\boldsymbol{M}^\alpha$, $\boldsymbol{m}$, g, G, h^α and H^α represent infinitesimal quantities of order $O(\varepsilon)$ and they are all zero in the reference configuration. Let

$$\boldsymbol{N}^\alpha = N^{\alpha i}\boldsymbol{A}_i, \qquad \boldsymbol{M}^\alpha = M^{\alpha i}\boldsymbol{A}_i, \qquad \boldsymbol{m} = m^i \boldsymbol{A}_i, \qquad \boldsymbol{f} = f^i \boldsymbol{A}_i, \qquad \boldsymbol{l} = l^i \boldsymbol{A}_i$$

and let $\rho_0 = \rho_0(\theta^\alpha)$ denote the mass density of the surface $\mathscr{S}_0$.

The linearized version of the equations of motion (2.16)–(2.18) can be written as follows:

$$N^{\alpha\beta}{}_{|\alpha} - B^\beta_\alpha N^{\alpha 3} + \rho_0 f^\beta = \rho_0 \ddot{u}^\beta, \qquad N^{\alpha 3}{}_{|\alpha} + B_{\alpha\beta} N^{\alpha\beta} + \rho_0 f^3 = \rho_0 \ddot{u}^3, \tag{3.5}$$

$$M^{\alpha\beta}{}_{|\alpha} - B^\beta_\alpha M^{\alpha 3} - m^\beta + \rho_0 l^\beta = \rho_0 \bar{\alpha} \ddot{\delta}^\beta, \qquad M^{\alpha 3}{}_{|\alpha} + B_{\alpha\beta} M^{\alpha\beta} - m^3 + \rho_0 l^3 = \rho_0 \bar{\alpha} \ddot{\delta}^3, \tag{3.6}$$

$$\epsilon_{\alpha\beta}(N^{\alpha\beta} - B^\alpha_\gamma M^{\gamma\beta}) = 0, \qquad N^{\alpha 3} - m^\alpha - B^\alpha_\gamma M^{\gamma 3} = 0, \tag{3.7}$$

where $\epsilon_{\alpha\beta}$ is the two-dimensional alternator defined by $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$.

The equations of equilibrated force (2.19) become

$$h^\alpha{}_{|\alpha} - g + \rho_0 p = \rho_0 \kappa_1 \ddot{\varphi}, \qquad H^\alpha{}_{|\alpha} - G + \rho_0 P = \rho_0 \kappa_2 \ddot{\psi}. \tag{3.8}$$

As a result of linearization, all tensors are now referred to the reference configuration and covariant differentiation is with respect to $A_{\alpha\beta}$. In view of the moment of momentum equation $(3.7)_1$, we can define the symmetric tensor $N'^{\alpha\beta}$ by

$$N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} + B^\beta_\gamma M^{\gamma\alpha}. \tag{3.9}$$

Using (2.21) and the invariance conditions under superposed rigid body motions, we obtain that the linear constitutive equations are

$$\begin{aligned} &\epsilon = \bar{\epsilon}\big(e_{\alpha\beta}, \gamma_i, \varkappa_{i\alpha}, \varphi, \psi, \varphi_{,\beta}, \psi_{,\beta}; A_{\alpha\beta}, B_{\alpha\beta}, v_0, \theta^\mu\big), \\ &N'^{\alpha\beta} = \rho_0 \frac{\partial \bar{\epsilon}}{\partial e_{\alpha\beta}}, \qquad M^{\alpha i} = \rho_0 \frac{\partial \bar{\epsilon}}{\partial \varkappa_{i\alpha}}, \qquad m^i = \rho_0 \frac{\partial \bar{\epsilon}}{\partial \gamma_i}, \\ &g = \rho_0 \frac{\partial \bar{\epsilon}}{\partial \varphi}, \qquad h^\alpha = \rho_0 \frac{\partial \bar{\epsilon}}{\partial \varphi_{,\alpha}}, \qquad G = \rho_0 \frac{\partial \bar{\epsilon}}{\partial \psi}, \qquad H^\alpha = \rho_0 \frac{\partial \bar{\epsilon}}{\partial \psi_{,\alpha}}. \end{aligned} \tag{3.10}$$

The partial derivative $\frac{\partial \bar{\epsilon}}{\partial e_{\alpha\beta}}$ is understood to have the symmetric form

$$\frac{1}{2}\left(\frac{\partial \bar{\epsilon}}{\partial e_{\alpha\beta}} + \frac{\partial \bar{\epsilon}}{\partial e_{\beta\alpha}}\right).$$

We notice that the components $N^{\alpha 3}$ can be determined from the moment of momentum equation $(3.7)_2$.

In the linear theory of Cosserat shells with the initial director given by (3.3) it is convenient to use the kinematic measures

$$\rho_{\beta\alpha} = \varkappa_{\beta\alpha} + B_{\alpha\beta}\gamma_3, \qquad \rho_{3\alpha} = \varkappa_{3\alpha} - B^{\beta}_{\alpha}\gamma_\beta, \tag{3.11}$$

and the variables

$$V^\alpha = N^{\alpha 3} = m^\alpha + B^{\alpha}_{\gamma} M^{\gamma 3}, \qquad V^3 = m^3 - B_{\alpha\beta} M^{\alpha\beta}. \tag{3.12}$$

In view of (3.4) and (3.11), in this case the geometrical relations are

$$\begin{aligned} &e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - B_{\alpha\beta}u_3, \qquad \gamma_\alpha = \delta_\alpha + u_{3,\alpha} + B^{\beta}_{\alpha}u_\beta, \\ &\gamma_3 = \delta_3, \qquad \rho_{\beta\alpha} = \delta_{\beta|\alpha} - B^{\gamma}_{\alpha}u_{\gamma|\beta} + B^{\gamma}_{\alpha}B_{\beta\gamma}u_3, \qquad \rho_{3\alpha} = \delta_{3,\alpha}. \end{aligned} \tag{3.13}$$

The equations of motion (3.5) and (3.6) can be written as

$$N^{\alpha\beta}{}_{|\alpha} - B^{\beta}_{\alpha}V^\alpha + \rho_0 f^\beta = \rho_0 \ddot{u}^\beta, \qquad V^\alpha{}_{|\alpha} + B_{\alpha\beta}N^{\alpha\beta} + \rho_0 f^3 = \rho_0 \ddot{u}^3, \tag{3.14}$$

$$M^{\alpha\beta}{}_{|\alpha} - V^\beta + \rho_0 l^\beta = \rho_0 \bar{\alpha} \ddot{\delta}^\beta, \qquad M^{\alpha 3}{}_{|\alpha} - V^3 + \rho_0 l^3 = \rho_0 \bar{\alpha} \ddot{\delta}^3. \tag{3.15}$$

By virtue of (3.10)–(3.12), the constitutive equations can be expressed in the form

$$\begin{aligned} &\epsilon = \tilde{\epsilon}\big(e_{\alpha\beta}, \gamma_i, \rho_{i\alpha}, \varphi, \psi, \varphi_{,\beta}, \psi_{,\beta}; A_{\alpha\beta}, B_{\alpha\beta}, v_0, \theta^\mu\big), \\ &N'^{\alpha\beta} = N'^{\beta\alpha} = \rho_0 \frac{\partial \tilde{\epsilon}}{\partial e_{\alpha\beta}}, \qquad V^i = \rho_0 \frac{\partial \tilde{\epsilon}}{\partial \gamma_i}, \qquad M^{\alpha i} = \rho_0 \frac{\partial \tilde{\epsilon}}{\partial \rho_{i\alpha}}, \\ &g = \rho_0 \frac{\partial \tilde{\epsilon}}{\partial \varphi}, \qquad h^\alpha = \rho_0 \frac{\partial \tilde{\epsilon}}{\partial \varphi_{,\alpha}}, \qquad G = \rho_0 \frac{\partial \tilde{\epsilon}}{\partial \psi}, \qquad H^\alpha = \rho_0 \frac{\partial \tilde{\epsilon}}{\partial \psi_{,\alpha}}. \end{aligned} \tag{3.16}$$

Here, $\tilde{\epsilon}$ is a quadratic function of the variables $e_{\alpha\beta}$, γ_i, $\rho_{i\alpha}$, φ, ψ, $\varphi_{,\beta}$ and $\psi_{,\beta}$, which coefficients depend on $\{A_{\alpha\beta}, B_{\alpha\beta}, v_0, \theta^\mu\}$.

The basic field equations of the linear theory of Cosserat shells with voids are the geometrical relations (3.13), the equations of motion (3.14) and (3.15), the equations of equilibrated force (3.8) and the constitutive relations (3.16). In order to formulate the boundary initial value problem, to the field equations we must adjoin boundary and initial conditions. Let $\mathscr{C}_i$ $(i = 1, \ldots, 6)$ be subsets of $\partial\mathscr{S}_0$ (assumed to be a piecewise smooth curve) such that

$$\overline{\mathscr{C}}_1 \cup \mathscr{C}_2 = \overline{\mathscr{C}}_3 \cup \mathscr{C}_4 = \overline{\mathscr{C}}_5 \cup \mathscr{C}_6 = \partial\mathscr{S}_0, \qquad \mathscr{C}_1 \cap \mathscr{C}_2 = \mathscr{C}_3 \cap \mathscr{C}_4 = \mathscr{C}_5 \cap \mathscr{C}_6 = \emptyset.$$

The boundary conditions are

$$\begin{aligned} &u_i = \tilde{u}_i \quad \text{on } \overline{\mathscr{C}}_1 \times \mathscr{T}, \qquad N^{\alpha i}n_\alpha = \widetilde{N}^i \quad \text{on } \mathscr{C}_2 \times \mathscr{T}, \\ &\delta_i = \tilde{\delta}_i \quad \text{on } \overline{\mathscr{C}}_3 \times \mathscr{T}, \qquad M^{\alpha i}n_\alpha = \widetilde{M}^i \quad \text{on } \mathscr{C}_4 \times \mathscr{T}, \\ &\varphi = \tilde{\varphi}, \ \ \psi = \tilde{\psi} \quad \text{on } \overline{\mathscr{C}}_5 \times \mathscr{T}, \qquad h^\alpha n_\alpha = \tilde{h}, \ \ H^\alpha n_\alpha = \widetilde{H} \quad \text{on } \mathscr{C}_6 \times \mathscr{T}, \end{aligned} \tag{3.17}$$

where $\mathscr{T} = [0, \infty)$ is the time interval and $\boldsymbol{n} = n_\alpha \boldsymbol{A}^\alpha$ is the outward unit normal to $\partial\mathscr{S}_0$, tangent to the surface $\mathscr{S}_0$.

We consider the following initial conditions, assumed to hold for each point θ^α

$$\begin{aligned} u_i(\theta^\alpha,0) &= u_{0i}(\theta^\alpha), & \dot{u}_i(\theta^\alpha,0) &= v_{0i}(\theta^\alpha),\\ \delta_i(\theta^\alpha,0) &= \delta_{0i}(\theta^\alpha), & \dot{\delta}_i(\theta^\alpha,0) &= w_{0i}(\theta^\alpha),\\ \varphi(\theta^\alpha,0) &= \varphi_0(\theta^\alpha), & \dot{\varphi}(\theta^\alpha,0) &= \lambda_0(\theta^\alpha),\\ \psi(\theta^\alpha,0) &= \psi_0(\theta^\alpha), & \dot{\psi}(\theta^\alpha,0) &= \mu_0(\theta^\alpha), \end{aligned} \tag{3.18}$$

where the functions on the right-hand sides of (3.18) are prescribed continuous functions on $\mathscr{S}_0$. We also assume that $\tilde{u}_i$, $\tilde{\delta}_i$, $\tilde{\varphi}$ and $\tilde{\psi}$ are continuous functions, while $\widetilde{N}^i$, $\widetilde{M}^i$, $\tilde{h}$ and $\widetilde{H}$ are continuous in time and piecewise regular on the appropriate domains.

We call *solution* of the boundary initial value problem a set of functions $\{u_i, \delta_i, \varphi, \psi\}$ of class C^1 on $\bar{\mathscr{S}}_0 \times \mathscr{T}$ and of class C^2 on $\mathscr{S}_0 \times \mathscr{T}$ such that they satisfy the system of field equations (3.8) and (3.13)–(3.16), the boundary conditions (3.17) and the initial conditions (3.18). In the remaining of this section, we shall prove a uniqueness result concerning this solution.

Let us denote by $K(t)$ the kinetic energy and by $U(t)$ the internal energy of the Cosserat shell, given by

$$K(t) = \frac{1}{2}\int_{\mathscr{S}_0} \rho_0(\boldsymbol{v}\cdot\boldsymbol{v} + \bar{\alpha}\boldsymbol{w}\cdot\boldsymbol{w} + \kappa_1\dot{\varphi}^2 + \kappa_2\dot{\psi}^2)\,\mathrm{d}a, \qquad U(t) = \int_{\mathscr{S}_0} \rho_0 \epsilon\,\mathrm{d}a, \tag{3.19}$$

where $\mathrm{d}a$ is the area element of the reference surface $\mathscr{S}_0$. Then, the balance of energy principle (2.9) for the Cosserat shell can be written in the linear theory in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[K(t) + U(t)] = \int_{\mathscr{S}_0} \rho_0\big(\boldsymbol{f}\cdot\boldsymbol{v} + \boldsymbol{l}\cdot\boldsymbol{w} + p\dot{\varphi} + P\dot{\psi}\big)\,\mathrm{d}a + \int_{\partial\mathscr{S}_0} \big(\boldsymbol{N}\cdot\boldsymbol{v} + \boldsymbol{M}\cdot\boldsymbol{w} + h\dot{\varphi} + H\dot{\psi}\big)\,\mathrm{d}l, \tag{3.20}$$

where $\mathrm{d}l$ is an element of length of the boundary curve $\partial\mathscr{S}_0$.

Remark 1. Using (3.20) we can prove by the same method as Naghdi (1972, Section 26), the uniqueness of the solution for the boundary initial value problem, in the hypotheses that the internal energy function $\tilde{\epsilon}$ is always nonnegative and $\rho_0 > 0$, $\bar{\alpha} > 0$, $\kappa_1 > 0$, $\kappa_2 > 0$.

Our purpose is to establish an uniqueness theorem without any definiteness assumptions on the internal energy function $\tilde{\epsilon}$. To this aim, let us prove the following result, which is a counterpart of Brun's theorem from the classical theory of elasticity (see, e.g., Gurtin, 1972, p. 217).

Theorem 1. *For any $t, s \in \mathscr{T}$, let*

$$\begin{aligned} Q(t,s) = &\int_{\mathscr{S}_0} \rho_0\big[\boldsymbol{f}(t)\cdot\boldsymbol{v}(s) + \boldsymbol{l}(t)\cdot\boldsymbol{w}(s) + p(t)\dot{\varphi}(s) + P(t)\dot{\psi}(s)\big]\,\mathrm{d}a\\ &+ \int_{\partial\mathscr{S}_0} \big[\boldsymbol{N}(t)\cdot\boldsymbol{v}(s) + \boldsymbol{M}(t)\cdot\boldsymbol{w}(s) + h(t)\dot{\varphi}(s) + H(t)\dot{\psi}(s)\big]\,\mathrm{d}l, \end{aligned} \tag{3.21}$$

where the dependence on the point θ^α of the above fields has been omitted for simplicity. Then, for every $t \in \mathscr{T}$, the following relation holds:

$$\begin{aligned} U(t) - K(t) = &\frac{1}{2}\int_0^t \big[Q(t+\tau, t-\tau) - Q(t-\tau, t+\tau)\big]\,\mathrm{d}\tau + \frac{1}{2}\int_{\mathscr{S}_0} \big[N'^{\alpha\beta}(0)e_{\alpha\beta}(2t) + V^i(0)\gamma_i(2t)\\ &+ M^{\alpha i}(0)\rho_{i\alpha}(2t) + g(0)\varphi(2t) + h^\alpha(0)\varphi_{,\alpha}(2t) + G(0)\psi(2t) + H^\alpha(0)\psi_{,\alpha}(2t)\big]\,\mathrm{d}a\\ &- \frac{1}{2}\int_{\mathscr{S}_0} \rho_0\big[\boldsymbol{v}(0)\cdot\boldsymbol{v}(2t) + \bar{\alpha}\boldsymbol{w}(0)\cdot\boldsymbol{w}(2t) + \kappa_1\dot{\varphi}(0)\dot{\varphi}(2t) + \kappa_2\dot{\psi}(0)\dot{\psi}(2t)\big]\,\mathrm{d}a. \end{aligned} \tag{3.22}$$

Proof. Taking into account that $\tilde{\epsilon}$ is a quadratic function, from the constitutive equations (3.16) it follows that

$$\begin{aligned}&\dot{N}'^{\alpha\beta}(t)e_{\alpha\beta}(s)+\dot{V}^i(t)\gamma_i(s)+\dot{M}^{\alpha i}(t)\rho_{i\alpha}(s)+\dot{g}(t)\varphi(s)+\dot{h}^\alpha(t)\varphi_{,\alpha}(s)+\dot{G}(t)\psi(s)+\dot{H}^\alpha(t)\psi_{,\alpha}(s)\\&\quad=N'^{\alpha\beta}(s)\dot{e}_{\alpha\beta}(t)+V^i(s)\dot{\gamma}_i(t)+M^{\alpha i}(s)\dot{\rho}_{i\alpha}(t)+g(s)\dot{\varphi}(t)+h^\alpha(s)\dot{\varphi}_{,\alpha}(t)+G(s)\dot{\psi}(t)\\&\qquad+H^\alpha(s)\dot{\psi}_{,\alpha}(t),\quad\forall t,s\in\mathscr{T}.\end{aligned}\tag{3.23}$$

For each $t\in\mathscr{T}$, let us define the function $E(\tau)$ by

$$\begin{aligned}E(\tau)&=N'^{\alpha\beta}(t-\tau)e_{\alpha\beta}(t+\tau)+V^i(t-\tau)\gamma_i(t+\tau)+M^{\alpha i}(t-\tau)\rho_{i\alpha}(t+\tau)+g(t-\tau)\varphi(t+\tau)\\&\quad+h^\alpha(t-\tau)\varphi_{,\alpha}(t+\tau)+G(t-\tau)\psi(t+\tau)+H^\alpha(t-\tau)\psi_{,\alpha}(t+\tau),\quad\tau\in[0,t].\end{aligned}\tag{3.24}$$

In view of (3.23) and (3.24), we have

$$\begin{aligned}\frac{dE}{d\tau}&=\big[N'^{\alpha\beta}(t-\tau)\dot{e}_{\alpha\beta}(t+\tau)+V^i(t-\tau)\dot{\gamma}_i(t+\tau)+M^{\alpha i}(t-\tau)\dot{\rho}_{i\alpha}(t+\tau)+g(t-\tau)\dot{\varphi}(t+\tau)\\&\quad+h^\alpha(t-\tau)\dot{\varphi}_{,\alpha}(t+\tau)+G(t-\tau)\dot{\psi}(t+\tau)+H^\alpha(t-\tau)\dot{\psi}_{,\alpha}(t+\tau)\big]-\big[N'^{\alpha\beta}(t+\tau)\dot{e}_{\alpha\beta}(t-\tau)\\&\quad+V^i(t+\tau)\dot{\gamma}_i(t-\tau)+M^{\alpha i}(t+\tau)\dot{\rho}_{i\alpha}(t-\tau)+g(t+\tau)\dot{\varphi}(t-\tau)+h^\alpha(t+\tau)\dot{\varphi}_{,\alpha}(t-\tau)\\&\quad+G(t+\tau)\dot{\psi}(t-\tau)+H^\alpha(t+\tau)\dot{\psi}_{,\alpha}(t-\tau)\big].\end{aligned}\tag{3.25}$$

On the other hand, by virtue of the relation (3.9), the geometrical relations (3.13) and the field equations (3.8), (3.14), (3.15) we can prove the following equality

$$\begin{aligned}&N'^{\alpha\beta}(t)\dot{e}_{\alpha\beta}(s)+V^i(t)\dot{\gamma}_i(s)+M^{\alpha i}(t)\dot{\rho}_{i\alpha}(s)+g(t)\dot{\varphi}(s)+h^\alpha(t)\dot{\varphi}_{,\alpha}(s)+G(t)\dot{\psi}(s)+H^\alpha(t)\dot{\psi}_{,\alpha}(s)\\&\quad=\big[N^{\alpha i}(t)\dot{u}_i(s)+M^{\alpha i}(t)\dot{\delta}_i(s)+h^\alpha(t)\dot{\varphi}(s)+H^\alpha(t)\dot{\psi}(s)\big]_{|\alpha}+\rho_0\big[f^i(t)\dot{u}_i(s)+l^i(t)\dot{\delta}_i(s)+p(t)\dot{\varphi}(s)\\&\qquad+P(t)\dot{\psi}(s)\big]-\rho_0\big[\ddot{u}^i(t)\dot{u}_i(s)+\bar{\alpha}\ddot{\delta}^i(t)\dot{\delta}_i(s)+\kappa_1\ddot{\varphi}(t)\dot{\varphi}(s)+\kappa_2\ddot{\psi}(t)\dot{\psi}(s)\big],\end{aligned}\tag{3.26}$$

for every $t,s\in\mathscr{T}$. If we integrate Eq. (3.25) over $\mathscr{S}_0$ and use relations of the type (3.26), then we obtain

$$\begin{aligned}\int_{\mathscr{S}_0}\frac{d}{d\tau}E(\tau)\,da&=Q(t-\tau,t+\tau)-Q(t+\tau,t-\tau)+\int_{\mathscr{S}_0}\rho_0\frac{d}{d\tau}\big[\boldsymbol{v}(t-\tau)\cdot\boldsymbol{v}(t+\tau)+\bar{\alpha}\boldsymbol{w}(t-\tau)\cdot\boldsymbol{w}(t+\tau)\\&\quad+\kappa_1\dot{\varphi}(t-\tau)\dot{\varphi}(t+\tau)+\kappa_2\dot{\psi}(t-\tau)\dot{\psi}(t+\tau)\big]\,da.\end{aligned}\tag{3.27}$$

Finally, by integrating (3.27) with respect to τ from 0 to t and using the relation $E(0)=2\rho_0\epsilon$, we deduce that (3.22) holds true. This completes the proof. □

On the basis of Theorem 1 and Eq. (3.20), we can prove the following uniqueness result.

Theorem 2. *Assume that the mass density ρ_0 and the inertia coefficients $\bar{\alpha}$, κ_1 and κ_2 are positive. Then, the boundary initial value problem (3.8), (3.13)–(3.18) has at most one solution.*

Proof. Suppose that the boundary initial value problem admits two solutions. Let us denote by $\{u_i,\delta_i,\varphi,\psi\}$ the difference of the two solutions. By the linearity of the theory, we see that $\{u_i,\delta_i,\varphi,\psi\}$ is a solution for the boundary initial value problem corresponding to null assigned body forces, null boundary conditions and null initial conditions. Then, in view of (3.20) and (3.22), we deduce

$$K(t)=0,\quad\forall t\in\mathscr{T}.\tag{3.28}$$

Since $\rho_0>0$, $\bar{\alpha}>0$, $\kappa_1>0$ and $\kappa_2>0$, the relation (3.28) yields

$$\boldsymbol{v}=\boldsymbol{w}=\boldsymbol{0},\qquad\dot{\varphi}=\dot{\psi}=0,$$

and, hence,

$$u_i = \delta_i = \varphi = \psi = 0.$$

The proof is complete. □

4. Isotropic and homogeneous materials

In this section, we specialize the linear theory of Cosserat shells and plates for the case of isotropic and homogeneous materials and we find explicit constitutive equations. Then, we present a comparison between the equations of Cosserat plates and the theory of plates derived from the three-dimensional equations for elastic materials with voids.

4.1. Cosserat shells and plates

Henceforth, we confine our attention to Cosserat surfaces possessing holohedral isotropy (i.e., isotropy with a center of symmetry). This material symmetry will place some restrictions on the constitutive equations. On the basis of arguments presented by Naghdi (1972) and Green and Naghdi (1979), we consider that the constitutive equations (3.16) are further restricted by the condition

$$\begin{aligned}&\tilde{\epsilon}\big(e_{\alpha\beta}, -\gamma_\alpha, \gamma_3, -\rho_{\beta\alpha}, \rho_{3\alpha}, \varphi, -\psi, \varphi_{,\beta}, -\psi_{,\beta}; A_{\alpha\beta}, -B_{\alpha\beta}, v_0, \theta^\mu\big)\\ &\quad = \tilde{\epsilon}\big(e_{\alpha\beta}, \gamma_\alpha, \gamma_3, \rho_{\beta\alpha}, \rho_{3\alpha}, \varphi, \psi, \varphi_{,\beta}, \psi_{,\beta}; A_{\alpha\beta}, B_{\alpha\beta}, v_0, \theta^\mu\big).\end{aligned} \tag{4.1}$$

Accordingly, the response function $\tilde{\epsilon}$ is assumed to be of the form

$$\begin{aligned}2\rho_0\tilde{\epsilon} = {}& \big[\alpha_1 A^{\alpha\beta}A^{\gamma\delta} + \alpha_2(A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})\big]e_{\alpha\beta}e_{\gamma\delta} + \alpha_3 A^{\alpha\beta}\gamma_\alpha\gamma_\beta + \alpha_4(\gamma_3)^2 + \big(\alpha_5 A^{\alpha\beta}A^{\gamma\delta} + \alpha_6 A^{\alpha\gamma}A^{\beta\delta}\\ &+ \alpha_7 A^{\alpha\delta}A^{\beta\gamma}\big)\rho_{\alpha\beta}\rho_{\gamma\delta} + \alpha_8 A^{\alpha\beta}\rho_{3\alpha}\rho_{3\beta} + 2\alpha_9 A^{\alpha\beta}e_{\alpha\beta}\gamma_3 + \beta_1 A^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + 2\beta_2 A^{\alpha\beta}\rho_{3\alpha}\varphi_{,\beta} + \beta_3\varphi^2\\ &+ 2\beta_4 A^{\alpha\beta}e_{\alpha\beta}\varphi + 2\beta_5\gamma_3\varphi + \beta_6\psi^2 + 2\beta_7 A^{\alpha\beta}\rho_{\alpha\beta}\psi + \beta_8 A^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} + 2\beta_9 A^{\alpha\beta}\gamma_\alpha\psi_{,\beta},\end{aligned} \tag{4.2}$$

where $\alpha_1, \ldots, \alpha_9$ and $\beta_1, \ldots, \beta_9$ are constant constitutive coefficients. From (3.16) and (4.2) we can write the explicit forms of the constitutive equations for $N'^{\alpha\beta}$, V^i, $M^{\alpha i}$, g, h^α, G and H^α. We mention that in the case when $\beta_k = 0$ $(k = 1, \ldots, 9)$ we obtain the constitutive equations for Cosserat shells (without voids).

In what follows, we shall record the full system of field equations for the case of Cosserat plates (i.e., initially flat Cosserat surfaces) in the linear theory and for isotropic materials.

The case of Cosserat plates is characterized by $B_{\alpha\beta} = 0$ and we remark that, in this situation, the boundary initial value problem separates into two systems of uncoupled equations: one system involves the kinematic variables u_α, δ_3, φ and represents the extensional motions (stretching), while the other involves the kinematic variables u_3, δ_α, ψ and describes the bending (or flexural motions) of the plate with voids.

In order to simplify the expressions, we choose the curvilinear coordinates (θ^α) on $\mathscr{S}_0$ to coincide with the coordinates (x_α) of the rectangular Cartesian frame $Ox_1x_2x_3$. Then, $A_{\alpha\beta} = \delta_{\alpha\beta}$ (the Kronecker symbol) and the covariant differentiation reduces to partial differentiation. The system of field equations uncouples as follows:

(i) *Extensional motion of a Cosserat plate*
- geometrical relations

$$e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \qquad \gamma_3 = \delta_3, \qquad \rho_{3\alpha} = \delta_{3,\alpha}; \tag{4.3}$$

- equations of motion

$$N_{\alpha\beta,\alpha} + \rho_0 f_\beta = \rho_0 \ddot{u}_\beta, \qquad M_{\alpha 3,\alpha} - V_3 + \rho_0 l_3 = \rho_0 \bar{\alpha} \ddot{\delta}_3; \tag{4.4}$$

- equation of equilibrated force

$$h_{\alpha,\alpha} - g + \rho_0 p = \rho_0 \kappa_1 \ddot{\varphi}; \tag{4.5}$$

- constitutive equations

$$\begin{aligned} N_{\alpha\beta} &= (\alpha_1 e_{\gamma\gamma} + \alpha_9 \gamma_3 + \beta_4 \varphi)\delta_{\alpha\beta} + 2\alpha_2 e_{\alpha\beta}, \\ M_{\alpha 3} &= \alpha_8 \rho_{3\alpha} + \beta_2 \varphi_{,\alpha}, \qquad V_3 = \alpha_4 \gamma_3 + \alpha_9 e_{\gamma\gamma} + \beta_5 \varphi, \\ g &= \beta_3 \varphi + \beta_4 e_{\gamma\gamma} + \beta_5 \gamma_3, \qquad h_\alpha = \beta_1 \varphi_{,\alpha} + \beta_2 \rho_{3\alpha}. \end{aligned} \tag{4.6}$$

(ii) *Bending of a Cosserat plate*

- geometrical relations

$$\gamma_\alpha = \delta_\alpha + u_{3,\alpha}, \qquad \rho_{\beta\alpha} = \delta_{\beta,\alpha}; \tag{4.7}$$

- equations of motion

$$V_{\alpha,\alpha} + \rho_0 f_3 = \rho_0 \ddot{u}_3, \qquad M_{\alpha\beta,\alpha} - V_\beta + \rho_0 l_\beta = \rho_0 \bar{\alpha} \ddot{\delta}_\beta; \tag{4.8}$$

- equation of equilibrated force

$$H_{\alpha,\alpha} - G + \rho_0 P = \rho_0 \kappa_2 \ddot{\psi}; \tag{4.9}$$

- constitutive equations

$$\begin{aligned} V_\alpha &= \alpha_3 \gamma_\alpha + \beta_9 \psi_{,\alpha}, \qquad M_{\alpha\beta} = (\alpha_5 \rho_{\gamma\gamma} + \beta_7 \psi)\delta_{\alpha\beta} + \alpha_6 \rho_{\alpha\beta} + \alpha_7 \rho_{\beta\alpha}, \\ G &= \beta_7 \rho_{\gamma\gamma} + \beta_6 \psi, \qquad H_\alpha = \beta_8 \psi_{,\alpha} + \beta_9 \gamma_\alpha. \end{aligned} \tag{4.10}$$

The equations of the extensional theory (4.3)–(4.6) and those of the bending theory (4.7)–(4.10) for Cosserat plates will be compared with the results obtained from the three-dimensional theory of elastic materials with voids in the next section.

4.2. Results from three-dimensional theory

The identification of the various tensors defined for Cosserat shells with voids (such as $N^{\alpha\beta}$, h^α, G, ...) with the corresponding resultants in the theory of shells derived from the three-dimensional equations of elastic materials with voids can be accomplished in the context of the nonlinear theory, regardless of any material symmetry assumptions, in the manner presented by Naghdi (1972) and Green and Naghdi (1979). Herein, for simplicity and for later reference, we restrict our attention to the case of linear plate theory for isotropic and homogeneous materials and we compare the equations obtained from the two different approaches.

Consider a three-dimensional body, made from an elastic material with voids, embedded in a Euclidean 3-space. The body is referred to a system of rectangular Cartesian axes Ox_i. Let u_i^* denote the displacement vector and e_{ij}^* be the strain tensor given by

$$e_{ij}^* = \tfrac{1}{2}(u_{i,j}^* + u_{j,i}^*). \tag{4.11}$$

The linear equations of motion are

$$t_{ji,j} + \rho_0^* f_i^* = \rho_0^* \ddot{u}_i^*, \tag{4.12}$$

where t_{ij} is the stress tensor, f_i^* the body forces per unit mass and ρ_0^* the reference mass density.

Let φ^* designate the change in volume fraction field, h_i^* the equilibrated stress, g^* the internal equilibrated body force and p^* the assigned equilibrated body force. Then, the local form of the balance law of equilibrated force is (see Cowin and Nunziato, 1983)

$$h_{i,i}^* - g^* + \rho_0^* p^* = \rho_0^* \kappa \ddot{\varphi}^*, \tag{4.13}$$

where κ represents the equilibrated inertia. The functions ρ_0^* and κ are constant and positive.

The constitutive equations for isotropic and homogeneous materials with voids are

$$t_{ij} = \lambda e_{rr}^* \delta_{ij} + 2\mu e_{ij}^* + b\varphi^* \delta_{ij}, \qquad h_i^* = \alpha \varphi_{,i}^*, \qquad g^* = b e_{rr}^* + \xi \varphi^*, \tag{4.14}$$

where λ, μ, b, α and ξ are constitutive constants.

We assume that the reference configuration of the body occupies the region

$$\mathscr{B} = \left\{ (x_1, x_2, x_3) \mid (x_1, x_2) \in \Sigma, -\frac{h_0}{2} < x_3 < \frac{h_0}{2} \right\},$$

where Σ is an open set in the $x_1 O x_2$ plane. The parameter h_0 is sufficiently small such that the body occupying $\mathscr{B}$ represents a plate of constant thickness h_0. Let $I = h_0^3/12$.

From Eqs. (4.12) and (4.13) we obtain, by integration with respect to x_3, the following relations

$$\widehat{N}_{\alpha\beta,\alpha} + \hat{f}_\beta = \hat{\rho}_0 \ddot{\hat{u}}_\beta, \qquad \widehat{M}_{\alpha 3,\alpha} - \widehat{V}_3 + \hat{l}_3 = \hat{\rho}_0 \hat{\alpha} \ddot{\hat{\delta}}_3, \qquad \hat{h}_{\alpha,\alpha} - \hat{g} + \hat{p} = \hat{\rho}_0 \kappa \ddot{\hat{\varphi}}, \tag{4.15}$$

and

$$\widehat{V}_{\alpha,\alpha} + \hat{f}_3 = \hat{\rho}_0 \ddot{\hat{u}}_3, \qquad \widehat{M}_{\alpha\beta,\alpha} - \widehat{V}_\beta + \hat{l}_\beta = \hat{\rho}_0 \hat{\alpha} \ddot{\hat{\delta}}_\beta, \qquad \widehat{H}_{\alpha,\alpha} - \widehat{G} + \widehat{P} = \hat{\rho}_0 \hat{\kappa} \ddot{\hat{\psi}}, \tag{4.16}$$

where we have introduced the notations

$$\begin{aligned}
&\hat{u}_i = \frac{1}{h_0} \int_{-h_0/2}^{h_0/2} u_i^* \,\mathrm{d}x_3, \qquad \hat{\delta}_i = \frac{1}{I} \int_{-h_0/2}^{h_0/2} x_3 u_i^* \,\mathrm{d}x_3, \qquad \hat{\varphi} = \frac{1}{h_0} \int_{-h_0/2}^{h_0/2} \varphi^* \,\mathrm{d}x_3, \\
&\hat{\psi} = \frac{1}{I} \int_{-h_0/2}^{h_0/2} x_3 \varphi^* \,\mathrm{d}x_3, \qquad \widehat{N}_{\alpha\beta} = \int_{-h_0/2}^{h_0/2} t_{\alpha\beta} \,\mathrm{d}x_3, \qquad \widehat{V}_i = \int_{-h_0/2}^{h_0/2} t_{3i} \,\mathrm{d}x_3, \\
&\widehat{M}_{\alpha i} = \int_{-h_0/2}^{h_0/2} x_3 t_{\alpha i} \,\mathrm{d}x_3, \qquad \hat{h}_\alpha = \int_{-h_0/2}^{h_0/2} h_\alpha^* \,\mathrm{d}x_3, \qquad \hat{g} = \int_{-h_0/2}^{h_0/2} g^* \,\mathrm{d}x_3, \\
&\widehat{H}_\alpha = \int_{-h_0/2}^{h_0/2} x_3 h_\alpha^* \,\mathrm{d}x_3, \qquad \widehat{G} = \int_{-h_0/2}^{h_0/2} x_3 g^* \,\mathrm{d}x_3 + \int_{-h_0/2}^{h_0/2} h_3^* dx_3,
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
&\hat{f}_i = \int_{-h_0/2}^{h_0/2} \rho_0^* f_i^* \,\mathrm{d}x_3 + 2t_{3i}^{(-)}\left(x_1, x_2, \frac{h_0}{2}, t\right), \qquad \hat{\rho}_0 = h_0 \rho_0^*, \\
&\hat{l}_i = \int_{-h_0/2}^{h_0/2} \rho_0^* x_3 f_i^* \,\mathrm{d}x_3 + h_0 t_{3i}^{(+)}\left(x_1, x_2, \frac{h_0}{2}, t\right), \qquad \hat{\alpha} = I/h_0, \\
&\hat{p} = \int_{-h_0/2}^{h_0/2} \rho_0^* p^* \,\mathrm{d}x_3 + 2h_3^{*(-)}\left(x_1, x_2, \frac{h_0}{2}, t\right), \qquad \hat{\kappa} = \kappa I/h_0, \\
&\widehat{P} = \int_{-h_0/2}^{h_0/2} \rho_0^* x_3 p^* \,\mathrm{d}x_3 + h_0 h_3^{*(+)}\left(x_1, x_2, \frac{h_0}{2}, t\right).
\end{aligned} \tag{4.18}$$

In (4.18) we have designated by $f^{(+)}$ and $f^{(-)}$ the expressions

$$f^{(+)}(x_1,x_2,x_3,t)=\tfrac{1}{2}\big[f(x_1,x_2,x_3,t)+f(x_1,x_2,-x_3,t)\big],$$
$$f^{(-)}(x_1,x_2,x_3,t)=\tfrac{1}{2}\big[f(x_1,x_2,x_3,t)-f(x_1,x_2,-x_3,t)\big],$$

for any function f defined on $\bar{\mathscr{B}}\times\mathscr{T}$.

Remark 2. In view of (4.11), (4.14), (4.17) and (4.18) we observe that Eqs. (4.15) involve only the variables $u_\alpha^{*(+)}$, $u_3^{*(-)}$ and $\varphi^{*(+)}$, while Eqs. (4.16) involve only the variables $u_\alpha^{*(-)}$, $u_3^{*(+)}$ and $\varphi^{*(-)}$. This property expresses the uncoupling between the extensional and bending deformations (see, e.g., Eringen, 1967, 1998). Thus, (4.15) represent the equations for the extensional motions, while (4.16) are the bending equations of the theory of plates with voids.

Basic assumptions of the plate theory allow a general representation as a polynomial in x_3 for the displacement vector u_i^* and for the volume fraction field φ^*, but we restrict attention to the approximation

$$u_i^*(x_1,x_2,x_3,t)=\hat{u}_i(x_1,x_2,t)+x_3\hat{\delta}_i(x_1,x_2,t),\qquad \varphi^*(x_1,x_2,x_3,t)=\hat{\varphi}(x_1,x_2,t)+x_3\hat{\psi}(x_1,x_2,t), \tag{4.19}$$

for every $(x_1,x_2,x_3,t)\in\bar{\mathscr{B}}\times\mathscr{T}$.

By virtue of (4.11) and (4.14)–(4.19), we obtain that the governing equations for plates with voids derived from the three-dimensional theory are of the same form as Eqs. (4.3)–(4.10) for Cosserat plates. In making such comparison, we identify $\hat{\rho}_0$, $\hat{u}_i$, $\hat{\delta}_i$, $\hat{\varphi}$, $\hat{\psi}$, $\widehat{N}_{\alpha\beta}$, $\widehat{V}_i$, $\widehat{M}_{\alpha i}$, $\hat{h}_\alpha$, $\hat{g}$, $\widehat{H}_\alpha$, $\widehat{G}$, respectively, with ρ_0, u_i, δ_i, φ, ψ, $N_{\alpha\beta}$, V_i, $M_{\alpha i}$, h_α, g, H_α, G and also

$$\hat{f}_i=\rho_0 f_i,\qquad \hat{l}_i=\rho_0 l_i,\qquad \hat{p}=\rho_0 p,\qquad \widehat{P}=\rho_0 P,\qquad \hat{\alpha}=\bar{\alpha},\qquad \kappa=\kappa_1,\qquad \hat{\kappa}=\kappa_2. \tag{4.20}$$

Remark 3. In view of the identifications $\varphi=\hat{\varphi}$, $\psi=\hat{\psi}$ and the relation $(4.19)_2$, we have

$$\varphi=\frac{1}{h_0}\int_{-h_0/2}^{h_0/2}\varphi^*\,\mathrm{d}x_3,\qquad \psi=\frac{1}{h_0}\int_{-h_0/2}^{h_0/2}\varphi^*_{,3}\,\mathrm{d}x_3.$$

Thus, we can interpret the fields φ and ψ associated to each point of the Cosserat surface as the average volume fraction and the average volume fraction gradient through the shell thickness, respectively. We see that φ enters the extensional problem (4.3)–(4.6), while ψ is involved only in the bending problem (4.7)–(4.10).

In the case when we do not take into account the variations of the volume fraction along the shell thickness, we obtain the theory of Cosserat shells with voids presented by Bîrsan (2000b, 2005) which accounts only for the changes in volume fraction along the middle surface of the shell. The equations of this theory can be derived from the developments of the present paper, by assuming that the internal energy response function ϵ do not depend on ψ and $\psi_{,\beta}$. (For the isotropic case, this is equivalent to put $\beta_6=\beta_7=\beta_8=\beta_9=0$ in the constitutive equations.)

Remark 4. We observe that the relations (4.19) are satisfied in the case of Mindlin-type plates. The equations of bending for elastic plates with voids obtained in this section have been studied recently by Scarpetta (2002) and Bîrsan (2003b, in press) in the context of the theory of Mindlin-type plates.

5. Determination of the constitutive coefficients

In the constitutive equations for Cosserat shells (3.16), (4.2) most of the constitutive coefficients $\alpha_1,\ldots,\alpha_9,\beta_1,\ldots,\beta_9$ can be identified by comparison of certain simple solutions with corresponding exact solutions in the three-dimensional theory of elastic materials with voids. The purpose of this section is to determine the constitutive coefficients by considering two elastostatic problems: the pure bending of a rectangular plate with voids and the extensional deformation of a plate subject to hydrostatic pressure. For the theory of Cosserat surfaces (without voids) the identification of constitutive coefficients has been discussed in several works (see, e.g., Naghdi, 1972, Section 24).

5.1. Pure bending of a plate

Consider a rectangular Cosserat plate with voids in equilibrium, whose reference configuration is bounded by the lines $x_1 = a_1, a_2$ and $x_2 = b_1, b_2$ in the x_1Ox_2 plane. We assume that the assigned body forces are null, i.e.,

$$f_3 = l_\beta = P = 0. \tag{5.1}$$

The plate is bent by uniform couples $\widetilde{M}_1$ and $\widetilde{M}_2$ acting along the edges $x_1 = \text{constant}$ and $x_2 = \text{constant}$, respectively, while the equilibrated stress vanishes on the edges. Thus, the boundary conditions are

$$\begin{aligned} &M_{11} = \widetilde{M}_1, \quad M_{12} = V_1 = 0, \quad H_1 = 0 \qquad \text{on } x_1 = a_1, a_2,\\ &M_{22} = \widetilde{M}_2, \quad M_{21} = V_2 = 0, \quad H_2 = 0 \qquad \text{on } x_2 = b_1, b_2, \end{aligned} \tag{5.2}$$

where $\widetilde{M}_\gamma$ $(\gamma = 1, 2)$ are prescribed constants. The relevant field equations for the bending problems of Cosserat plates with voids are (4.7)–(4.10). We search for a solution of these equations (in the case of equilibrium) such that

$$\gamma_\alpha = 0, \qquad \rho_{11} = C_1, \qquad \rho_{22} = C_2, \qquad \rho_{12} = 0, \qquad \psi = C_3, \tag{5.3}$$

where C_1, C_2, C_3 are some constants to be determined.
Using (4.7) and (5.3), we obtain

$$u_{3,11} = -C_1, \qquad u_{3,22} = -C_2, \qquad u_{3,12} = 0, \qquad \delta_\alpha = -u_{3,\alpha},$$

and, hence,

$$u_3 = -\tfrac{1}{2}(C_1x_1^2 + C_2x_2^2), \qquad \delta_1 = C_1x_1, \qquad \delta_2 = C_2x_2, \qquad \psi = C_3, \tag{5.4}$$

where we have neglected a rigid body displacement field of the Cosserat plate.
In view of (4.10) and (5.1), we see that the equilibrium equations corresponding to (4.8) are verified. Then, we can determine the constants C_i $(i = 1, 2, 3)$ from the equation of equilibrated force (4.9) and the boundary conditions (5.2). These relations reduce to the algebraic system

$$\begin{aligned} &(\alpha_5 + \alpha_6 + \alpha_7)C_1 + \alpha_5C_2 + \beta_7C_3 = \widetilde{M}_1,\\ &\alpha_5C_1 + (\alpha_5 + \alpha_6 + \alpha_7)C_2 + \beta_7C_3 = \widetilde{M}_2,\\ &\beta_7C_1 + \beta_7C_2 + \beta_6C_3 = 0. \end{aligned} \tag{5.5}$$

Thus, the solution of the problem considered is given by (5.4) and (5.5).
Consider now the corresponding problem for a three-dimensional rectangular plate made from an isotropic and homogeneous material with voids, which in its reference configuration occupies the region

$$\mathscr{B} = \left\{(x_1, x_2, x_3) \mid a_1 < x_1 < a_2, b_1 < x_2 < b_2, -\frac{h_0}{2} < x_3 < \frac{h_0}{2}\right\}. \tag{5.6}$$

The differential equations which govern the equilibrium of this plate are (4.11)–(4.14) (written for the static case) and we assume that the body forces f_i^* and p^* are null. The boundary conditions are the following:

(a) there are no stresses or equilibrated stresses acting on the top and bottom surfaces of the plate, i.e.,

$$t_{3i} = 0, \quad h_3^* = 0 \qquad \text{on } x_3 = \pm\frac{h_0}{2}. \tag{5.7}$$

(b) the normal stresses t_{11} and t_{22} acting on the edges $x_1 = a_1, a_2$ and $x_2 = b_1, b_2$, respectively, create (uniform) net bending moments $\widetilde{M}_1$ and $\widetilde{M}_2$, given by

$$\int_{-h_0/2}^{h_0/2} x_3 t_{11}\,\mathrm{d}x_3 = \widetilde{M}_1 \quad \text{on } x_1 = a_1, a_2, \qquad \int_{-h_0/2}^{h_0/2} x_3 t_{22}\,\mathrm{d}x_3 = \widetilde{M}_2 \quad \text{on } x_2 = b_1, b_2. \tag{5.8}$$

(c) except for the bending moments (5.8), there are no net forces, torques or bending moments acting on the edges $x_1 = a_1, a_2$ and $x_2 = b_1, b_2$ of the plate. Also, the equilibrated stress is null on the edges, i.e.,

$$h_1^* = 0 \quad \text{on } x_1 = a_1, a_2, \qquad h_2^* = 0 \quad \text{on } x_2 = b_1, b_2. \tag{5.9}$$

The problem formulated above, concerning the pure bending of a (three-dimensional) plate with voids, has been considered previously by Cowin and Nunziato (1983). They have obtained the following solution for this problem

$$\begin{aligned}
&u_1^* = \widehat{C}_1 x_1 x_3, \qquad u_2^* = \widehat{C}_2 x_2 x_3,\\
&u_3^* = -\frac{1}{2}(\widehat{C}_1 x_1^2 + \widehat{C}_2 x_2^2) + (\widehat{C}_1 + \widehat{C}_2)RH_0^2\left[\frac{1}{2}\left(1 - \frac{\sigma}{(1-\sigma)RH_0^2}\right)x_3^2 - l_0^2\left(\cosh\frac{x_3}{l_0}\right)\left(\cosh\frac{h_0}{2l_0}\right)^{-1}\right],\\
&\varphi^* = -(\widehat{C}_1 + \widehat{C}_2)RH_0\left[x_3 - l_0\left(\sinh\frac{x_3}{l_0}\right)\left(\cosh\frac{h_0}{2l_0}\right)^{-1}\right],
\end{aligned} \tag{5.10}$$

where $\widehat{C}_1$, $\widehat{C}_2$ are some constants, while σ, H_0, l_0 and R are constant expressions given in terms of the constitutive coefficients λ, μ, b, α and ξ by

$$\sigma = \frac{\lambda}{2(\lambda+\mu)}, \qquad H_0 = \frac{b}{\lambda + 2\mu}, \qquad \frac{1}{l_0^2} = \frac{\xi - bH_0}{\alpha}, \qquad R = \frac{2\mu l_0^2}{\alpha}. \tag{5.11}$$

The significance of these material parameters has been discussed by Cowin and Nunziato (1983) and Puri and Cowin (1985). Thus, σ corresponds to Poisson's ratio, l_0 is a material coefficient of dimension length and H_0, R are dimensionless numbers. It is shown that l_0 is always real and H_0, R are positive. Let us also introduce the geometric factor S given by (see Puri and Cowin, 1985)

$$S = 1 - \frac{12 l_0^3}{h_0^3}\left(\frac{h_0}{l_0} - 2\tanh\frac{h_0}{2l_0}\right) \quad (0 < S < 1). \tag{5.12}$$

We define the constant $\widehat{C}_3$ by the relation

$$\widehat{C}_3 = \frac{1}{I}\int_{-h_0/2}^{h_0/2} x_3\varphi^*\,\mathrm{d}x_3. \tag{5.13}$$

From the boundary conditions (5.8) and the relations $(5.10)_4$, (5.13), we obtain the following system for the determination of the constants $\widehat{C}_i$ $(i = 1, 2, 3)$

$$\begin{aligned}
&D\widehat{C}_1 + \sigma D\widehat{C}_2 + (1-\sigma)DH_0\widehat{C}_3 = \widetilde{M}_1,\\
&\sigma D\widehat{C}_1 + D\widehat{C}_2 + (1-\sigma)DH_0\widehat{C}_3 = \widetilde{M}_2,\\
&(1-\sigma)DH_0\widehat{C}_1 + (1-\sigma)DH_0\widehat{C}_2 + \frac{(1-\sigma)D}{RS}\widehat{C}_3 = 0.
\end{aligned} \tag{5.14}$$

We have employed the notations

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad C = \frac{Eh_0}{1 - \sigma^2}, \qquad D = \frac{I}{h_0}C, \tag{5.15}$$

where E represents Young's modulus and D is the flexural rigidity of the plate. We remark that the relations (5.10) yield

$$\frac{1}{h_0}\int_{-h_0/2}^{h_0/2} u_3^* \, dx_3 = -\frac{1}{2}(\widehat{C}_1 x_1^2 + \widehat{C}_2 x_2^2), \qquad \frac{1}{I}\int_{-h_0/2}^{h_0/2} x_3 u_\alpha^* \, dx_3 = \widehat{C}_\alpha x_\alpha \quad (\alpha \text{ not summed}), \tag{5.16}$$

where we have neglected a rigid body displacement field in writing $(5.16)_1$.

We are now in a position to compare the two solutions for the bending problem of a rectangular plate with voids obtained in the two different approaches. In line with the identifications made in Section 4.2, from the relations (4.17), (5.4), (5.13) and (5.16) we deduce that $C_i = \widehat{C}_i$ $(i = 1, 2, 3)$. Then, in view of the system of equations (5.5) and (5.14), the two solutions coincide provided we set

$$\alpha_5 + \alpha_6 + \alpha_7 = D, \qquad \alpha_5 = \sigma D, \qquad \beta_7 = (1 - \sigma)DH_0, \qquad \beta_6 = \frac{(1 - \sigma)D}{RS}. \tag{5.17}$$

Using the same arguments as Naghdi (1972), we choose $\alpha_6 = \alpha_7$ and, hence, from (5.17) we obtain

$$\alpha_5 = \sigma D, \qquad \alpha_6 = \alpha_7 = \frac{1 - \sigma}{2}D, \qquad \beta_6 = \frac{(1 - \sigma)D}{RS}, \qquad \beta_7 = (1 - \sigma)DH_0. \tag{5.18}$$

5.2. Extensional deformation of a plate subject to uniform hydrostatic pressure

Let us consider a rectangular Cosserat plate in equilibrium bounded by the lines $x_1 = a_1, a_2$ and $x_2 = b_1, b_2$. The plate suffers an extensional deformation due to constant forces per unit length $\widetilde{N}_1$ and $\widetilde{N}_2$ along the edges $x_1 = a_1, a_2$ and $x_2 = b_1, b_2$, respectively, and to a uniform hydrostatic pressure. We assume that $f_\beta = p = 0$. In view of (4.4) and (4.5), the equilibrium equations for this case are

$$N_{\alpha\beta,\alpha} = 0, \qquad M_{\alpha 3,\alpha} - V_3 + \rho_0 l_3 = 0, \qquad h_{\alpha,\alpha} - g = 0, \tag{5.19}$$

where $\rho_0 l_3$ is a constant which accounts for the uniform pressure prescribed on the surfaces of the plate and its significance will be clarified later in this section.

The appropriate boundary conditions are

$$\begin{aligned} &N_{11} = \widetilde{N}_1, \quad N_{12} = M_{13} = 0, \quad h_1 = 0 \qquad \text{on } x_1 = a_1, a_2,\\ &N_{22} = \widetilde{N}_2, \quad N_{21} = M_{23} = 0, \quad h_2 = 0 \qquad \text{on } x_2 = b_1, b_2. \end{aligned} \tag{5.20}$$

We search for a solution of this problem such that

$$e_{11} = K_1, \qquad e_{22} = K_2, \qquad e_{12} = 0, \qquad \gamma_3 = K_3, \qquad \varphi = K_4, \tag{5.21}$$

where K_i $(i = 1, \ldots, 4)$ are constants. Then, from the geometrical relations (4.3) we find that

$$u_1 = K_1 x_1, \qquad u_2 = K_2 x_2, \qquad \delta_3 = K_3, \qquad \varphi = K_4, \tag{5.22}$$

except for a rigid body displacement field. Using the constitutive relations (4.6) and imposing that the boundary conditions (5.20) and the equilibrium equations (5.19) be satisfied, we obtain that the constants K_i $(i = 1, \ldots, 4)$ verify the algebraic system of equations

$$\begin{aligned} &(\alpha_1 + 2\alpha_2)K_1 + \alpha_1 K_2 + \alpha_9 K_3 + \beta_4 K_4 = \widetilde{N}_1, \qquad \alpha_1 K_1 + (\alpha_1 + 2\alpha_2)K_2 + \alpha_9 K_3 + \beta_4 K_4 = \widetilde{N}_2,\\ &\alpha_9 K_1 + \alpha_9 K_2 + \alpha_4 K_3 + \beta_5 K_4 = \rho_0 l_3, \qquad \beta_4 K_1 + \beta_4 K_2 + \beta_5 K_3 + \beta_3 K_4 = 0. \end{aligned} \tag{5.23}$$

Relations (5.22) and (5.23) give the solution of the problem.

In what follows we consider the same problem formulated for a three-dimensional plate of constant thickness h_0. Thus, we study the equilibrium of a plate with voids which occupies the region (5.6) in its reference configuration. The body forces f_i^*, p^* are vanishing and the boundary conditions are specified by

$$\begin{aligned} &t_{11} = \widetilde{N}_1/h_0, \quad t_{12} = t_{13} = 0, \quad h_1 = 0 \qquad \text{on } x_1 = a_1, a_2, \\ &t_{22} = \widetilde{N}_2/h_0, \quad t_{21} = t_{23} = 0, \quad h_2 = 0 \qquad \text{on } x_2 = b_1, b_2, \\ &t_{33} = -p_0, \quad t_{31} = t_{32} = 0, \quad h_3 = 0 \qquad \text{on } x_3 = \pm h_0/2, \end{aligned} \tag{5.24}$$

where p_0 designates the uniform hydrostatic pressure acting on the top and bottom surfaces of the plate. The field equations are (4.11)–(4.14), written for the static case.

We search for the solution of this elastostatic problem in the form

$$u_i^* = \widehat{K}_i x_i \quad (i \text{ not summed}), \qquad \varphi^* = \widehat{K}_4, \tag{5.25}$$

where $\widehat{K}_i$ $(i = 1,\ldots,4)$ are some constants. By virtue of (4.11) and (4.14), we obtain that the boundary conditions (5.24) and the equilibrium equations reduce to the relations

$$\begin{aligned} &(\lambda + 2\mu)\widehat{K}_1 + \lambda\widehat{K}_2 + \lambda\widehat{K}_3 + b\widehat{K}_4 = \widetilde{N}_1/h_0, \qquad \lambda\widehat{K}_1 + (\lambda + 2\mu)\widehat{K}_2 + \lambda\widehat{K}_3 + b\widehat{K}_4 = \widetilde{N}_2/h_0, \\ &\lambda\widehat{K}_1 + \lambda\widehat{K}_2 + (\lambda + 2\mu)\widehat{K}_3 + b\widehat{K}_4 = -p_0, \qquad b\widehat{K}_1 + b\widehat{K}_2 + b\widehat{K}_3 + \xi\widehat{K}_4 = 0, \end{aligned} \tag{5.26}$$

which allow us to determine the constants $\widehat{K}_i$ $(i = 1,\ldots,4)$. For the solution (5.25), we have

$$\begin{aligned} &\frac{1}{h_0}\int_{-h_0/2}^{h_0/2} u_\alpha^* \,\mathrm{d}x_3 = \widehat{K}_\alpha x_\alpha \quad (\alpha \text{ not summed}), \\ &\frac{1}{I}\int_{-h_0/2}^{h_0/2} x_3 u_3^* \,\mathrm{d}x_3 = \widehat{K}_3, \qquad \frac{1}{h_0}\int_{-h_0/2}^{h_0/2} \varphi^* \,\mathrm{d}x_3 = \widehat{K}_4. \end{aligned} \tag{5.27}$$

We are now able to compare the two solutions obtained in the two different approaches of the problem considered. In view of the identifications made in Section 4.2 and the relations $(4.17)_{1\text{–}3}$, (5.22) and (5.27), we derive that $K_i = \widehat{K}_i$ $(i = 1,\ldots,4)$. Using $(4.18)_2$, $(4.20)_2$ and the boundary conditions (5.24) we see that, in our case, we have

$$\rho_0 l_3 = -h_0 p_0. \tag{5.28}$$

Hence, by comparison between the systems (5.23) and (5.26), we may identify

$$\alpha_1 + 2\alpha_2 = \alpha_4 = (\lambda + 2\mu)h_0, \qquad \alpha_1 = \alpha_9 = \lambda h_0, \qquad \beta_4 = \beta_5 = bh_0, \qquad \beta_3 = \xi h_0. \tag{5.29}$$

Using the notations (5.15), the relations (5.29) can be written as

$$\alpha_1 = \alpha_9 = \frac{\sigma(1-\sigma)}{1-2\sigma}C, \qquad \alpha_2 = \frac{1-\sigma}{2}C, \qquad \alpha_4 = \frac{(1-\sigma)^2}{1-2\sigma}C, \qquad \beta_3 = \xi h_0, \qquad \beta_4 = \beta_5 = bh_0. \tag{5.30}$$

This completes our consideration of simple elastostatic problems for plates with voids. To recapitulate, in (5.18) and (5.30) we have determined the constitutive coefficients α_1, α_2, α_4, α_5, α_6, α_7, α_9 and β_3, β_4, β_5, β_6, β_7 . We mention that the values for the coefficients α_k obtained in $(5.18)_{1,2}$ and $(5.30)_{1\text{–}3}$ coincide with the identification of the constitutive coefficients given by Naghdi (1972) for Cosserat surfaces without voids.

Two other coefficients, namely β_2 and β_9, can be identified by comparison between the constitutive equations (4.6), (4.10) and the relations obtained by substituting (4.11), (4.14) and (4.19) into (4.17). This procedure suggests the values

$$\beta_2 = 0, \qquad \beta_9 = 0. \tag{5.31}$$

The constitutive coefficients α_3, α_8, β_1 and β_8 remain unspecified and they have the orders of magnitude

$$\alpha_3 = O(C), \qquad \alpha_8 = O(D), \qquad \beta_1 = O(C), \qquad \beta_8 = O(D). \tag{5.32}$$

Similar arguments as those presented by Naghdi (1972, Section 24), suggest that it is preferable to allow these coefficients to have different possible values, depending on the particular context in which the theory of Cosserat shells is used.

Finally we remark that, although the values of α_k and β_k ($k = 1, \ldots, 9$) have been determined for Cosserat plates, the identification of the constitutive coefficients (5.18), (5.30)–(5.32) is also valid for shells. Indeed, this fact can be deduced from the constitutive equations for Cosserat shells (3.16), (4.2) and the developments of Section 4.1.

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